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THE LATTICE PROPERTIES OF VECTOR LATTICES

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UNDERGRADUATE THESIS

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CONTENTS

LIST OF SYMBOLS

E^+	The set of positive vectors of E
infA	The infimum of set A
supA	The supremum of set A
$x \vee y$	The supremum of the set {x,y}
$x \wedge y$	The infimum of the set {x,y}
x^+	The positive part of vector x
x^-	The negative part of vector x
$\lvert x \rvert$	The absolute value(or modulus) of vector x
α^-	Inverse of α
$x \perp y$	The orthogonality of vectors x and y
$\mathbb{R}$	The set of real numbers
Sol(A)	Solid hull of set A
$x_\alpha \uparrow$	The increasing net
$x_\alpha \downarrow$	The decreasing net
$x_\alpha \to x$	Order convergence of x_α
E_A	The ideal generated by set A
E_x	The ideal generated by vector x
A^d	Disjoint complement of set A
$\mathbb{N}$	The set of natural numbers
Ker(T)	Kernel of set T
T^{-1}	Inverse of set operator T
[x,y]	Order interval

LIST OF ABBREVIATION

AMS	American Mathematical Society

ABSTRACT

This thesis focus on the lattice properties of vector spaces. First of all, to help our subject, we examined what Riesz space is and some of its properties. Then, we showed some lattice properties and from these properties we proved the widely used theorems in Riesz spaces. Later, we examined some important definitions such as ideal, band in Riesz space and from here we showed some theorems and lemmas that were proved based on those definitions. Finally, we end the thesis with a short section about order completeness and Riesz algebra.

Key Words: Lattice, Ideal, Identities

1. THE LATTİCE STRUCTURE OF RİESZ SPACES

1.1. Elementary Properties of Riesz Spaces

A real vector space E with an order relation ≤ (also known as a reflexive, antisymmetric, transitive relation ≤) that is compatible with the algebraic structure of E in the sense that it has the following two properties is referred to as an ordered vector space:

(i) If $x \leq y$, then $x + z \leq y + z$ holds for every $z \in E$.

(ii)) If $x \leq y$, then $\alpha x \leq \alpha y$ holds for each $\alpha \geq 0$.

The zero element of a vector space will be represented by the number 0. The elements x of *E* is called positive vectors of *E* whenever $x \geq 0$. The set of all positive vectors of E will be denoted by E^+. $E^+ = \{ x \in E : x \geq 0 \}$ is called a cone (or positive cone) of *E*.

A nonempty subset *A* of *E* is said to have a supremum if there is some $x \in E$ that satisfies a $\leq x$ for all $a \in A$ and such that whenever $a \leq y$ holds for any $y \in E$ and all $a \in A$, then $x \leq y$. The supremum of *A* (or the least upper bound of *A*) is defined by sup*A* . The infimum of *A* (or greatest lower bound of *A*) is defined by inf*A*, has a similar definition.

Definition 1.1.1. An ordered vector space *E* is called a Riesz space(or vector lattice) if $\sup\{x,y\}$ and $\inf\{x,y\}$ both exist in *E* for every $x,y \in E$.

We will use $x \vee y$ to represent the supremum of the set $\{x,y\}$. Similarly, $x \wedge y$ represents the infimum of the set $\{x,y\}$. That means $\sup\{x,y\} = x \vee y$ and $\inf\{x,y\} = x \wedge y$.

Let $x \in E$ be a Riesz space then the positive part of x is defined by $x^+= x \vee 0$, the negative part of x is defined by $x^-= (-x) \vee 0$, the absolute value(or modulus) of x defined by $|x|= (-x) \vee x$.

Theorem 1.1.2. (Lattice Identities). Let E be a Riesz space and x, y, $z \in E$. Then, the following identities are true:

(1) $x \vee y = -[(-x)\wedge(-y)]$

(2) $x \wedge y = -[(-x)\vee(-y)]$

(3) $x + (y\vee z) = (x + y)\vee(x + z)$ and $x + (y\wedge z) = (x + y)\wedge(x + z)$

(4) $x - (y\vee z) = (x - y)\vee(x - z)$ and $x - (y\wedge z) = (x - y)\wedge(x - z)$

(5) $x\vee y = (x - y)^{+}+ y = (y - x)^{+} + x$

(6) $\alpha(x\vee y) = (\alpha x)\vee(\alpha y)$ and $\alpha(x\wedge y) = (\alpha x)\wedge(\alpha y)$ for all $\alpha \geq 0$

(7) $|\alpha x| = |\alpha||x|$ for all $\alpha \in \mathbb{R}$

(8) $x\vee y = \frac{1}{2}(x + y + |x - y|)$ and $x\wedge y = \frac{1}{2}(x + y - |x - y|)$

(9) $x + y = x\vee y + x\wedge y$

(10) $x = x^{+} - x^{-}$ and $x^{+}\wedge x^{-}= 0$

(11) $|x| = x^{+} + x^{-}$

(12) $|x - y| = x\vee y - x\wedge y$

(13) $|x + y|\vee|x - y| = |x| + |y|$

(14) $|x|\vee|y| = \frac{1}{2}(|x + y| + |x - y|)$ and $|x|\wedge|y| = \frac{1}{2}(|x + y| - |x - y|)$

Proof. (1) We know that $x \leq x\vee y$ and $y \leq x\vee y$. Multiply with (-1) both side these equations and the we get $-x \geq -(x\vee y)$, $-y \geq -(x\vee y)$. That means $(-x)\wedge(-y) \geq -(x\vee y)$. On the other hand, assume that $-x \geq z$, $-y \geq z$. Then $x \leq -z$, $y \leq -z$. From here, we get $-z$ is an upper bound of the set $\{x,y\}$ such that $x\vee y \leq -z$(or $z \leq -(x\vee y)$). This shows that $-(x\vee y)$ is the infimum of the set $\{-x,-y\}$. So, $(-x)\wedge(-y) = -(x\vee y)$ or $x\vee y = -[(-x)\wedge(-y)]$.

(2) This identity can be proved similarly by looking at the proof in (1).

(3) Let $a = y\vee z$. We know that $y \leq a$ and $z \leq a$. Clearly, $x + y \leq x + a$ and $x + z \leq x + a$. From here, we get $(x + y)\vee(x + z) \leq x + y\vee z$. Conversely, let $b = (x + y)\vee(x + z)$. Then, $x + y \leq b$ and $x + z \leq b$. From here we get $y \leq (-x) + b$ and $z \leq (-x) + b$. This shows that $y\vee z \leq (-x) + (x + y)\vee(x + z)$ or $x + y\vee z \leq (x + y)\vee(x + z)$. From inequalities, we get $x + (y\vee z) = (x + y)\vee(x + z)$. The other identity can be proven in a similiar manner.

(4) The validity of these identities can be made from the proof (1), (2) and (3).

(5) We will use equation $x^+ = x \vee 0$ to prove the identity

$(x - y)^+ + y = (x - y) \vee 0 + y = [(x - y) + y] \vee (0 + y) = x \vee y$

$(y - x)^+ + x = (y - x) \vee 0 + x = [(y - x) + x] \vee (0 + x) = y \vee x = x \vee y$

(6) For $\alpha = 0$, it is clear. Assume that $\alpha \geq 0$. We know that $x \leq x \vee y$ and $y \leq x \vee y$. From here, we get $\alpha x \leq \alpha(x \vee y)$ and $\alpha y \leq \alpha(x \vee y)$. That means $(\alpha x) \vee (\alpha y) \leq \alpha(x \vee y)$. Then assume that $z \geq \alpha x$ and $z \geq \alpha y$. It follows that $\alpha^{-1} z \geq x$ and $\alpha^{-1} z \geq y$. Hence we get $\alpha^{-1} z \geq x \vee y$ or $z \geq \alpha(x \vee y)$. This shows that $\alpha(x \vee y)$ is the supremum of the set $\{\alpha x, \alpha y\}$. Consequently we get $\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$. A similar method can be used to reveal the second identity.

(7) To prove this identity we will use $|x| = x \vee (-x)$.

For $\alpha \geq 0$, we get $|\alpha x| = (\alpha x) \vee (-\alpha x) = \alpha[x \vee (-x)] = |\alpha||x|$.

For $\alpha < 0$, we get $|\alpha x| = (\alpha x) \vee (-\alpha x) = [(-\alpha)(-x)] \vee (-\alpha x) = (-\alpha)[(-x) \vee x] = |\alpha||x|$.

(8) We will use $|x| = x \vee (-x)$ to prove the first identity.

$x + y + |x - y| = x + y + (x - y) \vee (y - x) = (x + y + x - y) \vee (x + y + y - x) = (2x) \vee (2y) = 2(x \vee y)$. Hence, we get $x \vee y = \frac{1}{2}(x + y + |x - y|)$. The other identity can be proved in a similar manner.

(9) The identity is obtained by adding the identities in (8).

(10) For the first identity we will use (9), then we get,

$x = x + 0 = x \vee 0 + x \wedge 0 = x \vee 0 - (-x) \vee 0 = x - x^-$. For the second identity we will use the first identity, then we get $x^+ \wedge x^- = (x^+ - x^-) \wedge 0 + x^- = x \wedge 0 + x^- = -[(-x) \vee 0] + x^- = (-x^-) + x^- = 0$.

(11) Using the definition of absolute value, then we get

$$|x| = x \vee (-x) = (2x) \vee 0 - x = 2(x \vee 0) - x = 2x^+ - (x^+ - x^-) = x^+ + x^-.$$

(12) From (8) we get

$$\begin{aligned} x \vee y - x \wedge y &= \frac{1}{2}(x + y + |x - y|) - \frac{1}{2}(x + y - |x - y|) \\ &= \frac{1}{2}(x + y + |x - y| - x - y + |x - y|) = |x - y|. \end{aligned}$$

(13) Observe that $|x+y| \vee |x-y| = [(x+y) \vee (-x-y)] \vee [(x-y) \vee (y-x)]$

$$= [(x+y) \vee (x-y)] \vee [(-x-y) \vee (y-x)]$$

$$= [x + y \vee (-y)] \vee [-x + (-y) \vee y]$$

$$= [x + |y|] \vee [-x + |y|]$$

$$= [x \vee (-x)] + |y|$$

$$= |x| + |y|.$$

(14) For the first identity we will use (8), then we get

$$|x+y| + |x-y| = (x+y) \vee (-x-y) + |x-y|$$

$$= (x + y + |x-y|) \vee (-x - y + |x-y|)$$

$$= 2([x \vee y] \vee [(-x) \vee (-y)])$$

$$= 2([x \vee (-x)] \vee [y \vee (-y)])$$

$$= 2(|x| \vee |y|).$$

From here $|x| \vee |y| = \frac{1}{2}(|x+y| + |x-y|)$.

For the second identity we will use (9), (12), and (13), then we get

$$|x+y| - |x-y| = |x+y| \vee |x-y| - |x+y| \wedge |x-y|$$

$$= |x+y| \vee |x-y| - (|x+y| + |x-y| - |x+y| \vee |x-y|)$$

$$= 2(|x+y| \vee |x-y|) - (|x+y| + |x-y|)$$

$$= 2(|x| + |y|) - 2(|x| \vee |y|)$$

$$= 2(|x| \wedge |y|).$$

From here $|x| \wedge |y| = \frac{1}{2}(|x+y| - |x-y|)$. ■

Lemma 1.1.3 (The Infinite Distributive Law). Let *A* be a nonempty subset of a Riesz space *E*. If sup*A* exists, then for every vector $x \in E$ there is $\sup\{ x \wedge a : a \in A \}$ and $x \wedge \sup A = \sup\{ x \wedge a : a \in A \}$. Similarly, if infA exists, then for every vector $x \in E$ there is $\inf\{ x \vee a : a \in A \}$ and $x \vee \inf A = \inf\{ x \vee a : a \in A \}$.

Proof. Assume that sup*A* exists. Let $y = \sup A$ and $x \in E$. Then, $x \wedge a \le x \wedge y$ for every $a \in A$. From here, let $b \in E$ be an upper bound of the set $x \wedge A = \{ x \wedge a : a \in A \}$. This means $x \wedge a \le b$ holds for every $a \in A$. We know that $x + y = x \vee y + x \wedge y$ is true in Riesz spaces. Using this lattice identity, we get $x + a - x \vee a = x \wedge a \le b$ for every $a \in A$. From here we have $a \le b + x \vee a - x \le b + x \vee y - x$

for every $a \in A$. Then, $y \leq b + x \vee y - x$. From here we get $x \wedge \sup A = \sup \{ x \wedge a : a \in A \}$. Other formula can be proven in this way. ■

We will present a classical identity in Riesz spaces known as Birkhoff's identity which is a direct application of the distrubituve laws.

Corollary 1.1.4 (Birkhoff's Identity). Let E be a Riesz space and $x, y, z \in E$. Then,

$$|x \vee z - y \vee z| + |x \wedge z - y \wedge z| = |x - y| \text{ holds.}$$

Proof. To prove this identity we will use the lattice identities (9) and (12) of Theorem 1.1.2. with the distributive laws, then we get

$$\begin{aligned} |x \vee z - y \vee z| + |x \wedge z - y \wedge z| &= [(x \vee z) \vee (y \vee z) - (x \vee z) \wedge (y \vee z)] + [(x \wedge z) \vee (y \wedge z) - (x \wedge z) \wedge (y \wedge z)] \\ &= [z \vee (x \vee y) - z \vee (x \wedge y)] + [z \wedge (x \vee y) - z \wedge (x \wedge y)] \\ &= [z \vee (x \vee y) + z \wedge (x \vee y)] - [z \vee (x \wedge y) + z \wedge (x \wedge y)] \\ &= [z + x \vee y] - [z + x \wedge y] \\ &= x \vee y - x \wedge y \\ &= |x - y|. \end{aligned}$$ ■

Theorem 1.1.5 (Lattice Inequalities). In Riesz spaces the following lattice inequalities are true.

(1) (The Triangle Inequality) Let x and y be arbitrary vectors in a Riesz space, then

$$||x| - |y|| \leq |x + y| \leq |x| + |y|$$

(2) (Birkhoff's Inequalities) Let x, y, and z be arbitrary vevtors in a Riesz space, then

$$|x \vee z - y \vee z| \leq |x - y| \text{ and } |x \wedge z - y \wedge z| \leq |x - y|$$

(3) Let x and y be arbitrary vectors in a Riesz space satisfy $x \leq y$, then

$$x^+ \leq y^+ \text{ and } y^- \leq x^-.$$

(4) Let $x, x_1, x_2, \dots, x_n$ be positive vectors in a Riesz space, then

$$x \wedge (x_1 + x_2 + \dots + x_n) \leq x \wedge x_1 + x \wedge x_2 + \dots + x \wedge x_n.$$

(5) Let $x_1, x_2, \dots, x_n$ be vectors in a Riesz space, then

$$n(x_1^+ \wedge \dots \wedge x_n^+) = n(x_1 \wedge \dots \wedge x_n)^+ \leq (x_1 + \dots + x_n)^+.$$

Proof. **(1)** Clearly, $x + y \leq |x| + |y|$ and $-x - y = -(x + y) \leq |x| + |y|$. From here, we get $|x + y| = (x + y) \vee [-(x + y)] \leq |x| + |y|$. For the other side, we have

$|x| = |(x + y) - y| \leq |x + y| + |y|$ and hence we get $|x| - |y| \leq |x + y|$. Also we get $|y| = |(y + x) + (-x)| \leq |y + x| + |-x| = |y + x| + |x|$ or $|y| - |x| = -(|x| - |y|) \leq |y + x| = |x + y|$. So $||x| - |y|| \leq |x + y|$. Thus, $||x| - |y|| \leq |x + y| \leq |x| + |y|$ is true.

(2) Notice that

$$
\begin{aligned}
x\vee z - y\vee z &= (x - z)\vee 0 + z - (y\vee z) \\
&= (x - z)^+ + z + [(-y)\wedge(-z)] \\
&= (x - z)^+ + (z-y)\wedge 0 \\
&= (x - z)^+ - [(y - z)\vee 0] \\
&= (x - z)^+ - (y - z)^+ \\
&= [(x - y) + (y - z)]^+ - (y - z)^+ \\
&\leq (x - y)^+ + (y - z)^+ - (y - z)^+ \\
&= (x - y)^+ \\
&\leq |x - y|
\end{aligned}
$$

$$
\begin{aligned}
y\vee z - x\vee z &= (y - z)\vee 0 + z - (x\vee z) \\
&= (y - z)^+ + z + [(-x)\wedge(-z)] \\
&= (y - z)^+ + (z-x)\wedge 0 \\
&= (y - z)^+ - [(x - z)\vee 0] \\
&= (y - z)^+ - (x - z)^+ \\
&= [(y - x) + (x - z)]^+ - (x - z)^+ \\
&\leq (y - x)^+ + (x - z)^+ - (x - z)^+ \\
&= (y - x)^+ \\
&\leq |y - x| = |x - y|
\end{aligned}
$$

So, $|x\vee z - y\vee z| \leq |x - y|$. The other inequality can be proven in a similar way.

(3) Let $x \leq y$. In that case $x \leq y \leq y\vee 0 = y^+$ and $0 \leq y^+$. From here $x^+ = x\vee 0 \leq y^+$. For other inequality we have $-y \leq -x$. Then $-y \leq -x \leq (-x)\vee 0 = x^-$ and $0 \leq x^-$. From here $y^- = (-y)\vee 0 \leq x^-$. The proof of these inequalities is finished.

(4) Let $y = x\wedge(x_1 + x_2)$. Then $y \leq x_1 + x_2$ and from here we have $y - x_1 \leq x_2$. Likewise we get $y - x_1 \leq y \leq x_2$. So $y - x_1 \leq x\wedge x_2$. This means $y - x\wedge x_2 \leq x_1$ and hence $y - x\wedge x_2 \leq y \leq x$. From the inequality $y - x\wedge x_2 \leq x\wedge x_1$ or $y \leq x\wedge x_1 + x\wedge x_2$. Put y in the equality then we have

$x\wedge(x_1 + x_2) \leq x\wedge x_1 + x\wedge x_2$. The general case can be proven by induction.

(5) From this inequality $n(x_1\wedge \ldots \wedge x_n) \leq x_1 + \ldots + x_n$ we can get $n(x_1\wedge \ldots \wedge x_n)^+ \leq (x_1 + \ldots + x_n)^+$. For other side we have $n(x_1^+\wedge \ldots \wedge x_n^+) = n[(x_1\vee 0) \wedge (x_2\vee 0) \wedge \ldots \wedge (x_n\vee 0)]$

$$= n[(x_1\wedge x_2\wedge \ldots \wedge x_n)\vee 0]$$

$$= n(x_1\wedge x_2\wedge \ldots \wedge x_n)^+.$$

So, we get $n(x_1^+\wedge \ldots \wedge x_n^+) = n(x_1\wedge \ldots \wedge x_n)^+ \leq (x_1 + \ldots + x_n)^+$. ■

Definition 1.1.6. Let *E* be a Riesz space and $x, y \in E$. x and y are called disjoint (or orthogonal) if $|x|\wedge|y| = 0$. The symbol $x\perp y$ represents orthogonality of x and y.[$x\perp y \Leftrightarrow |x|\wedge|y| = 0$]

Let *A* and *B* be nonempty subsets of a Riesz space. A and B are called disjoint(or orthogonal) if $|a|\wedge|b| = 0$ for all $a \in A$ and $b \in B$. The symbol $A\perp B$ represents orthogonality of *A* and *B*.

Lemma 1.1.7 (Disjointness Properties). Let *E* be a Riesz space and $x, y, z \in E$. Then the followings are true:

(1) If $x\perp y$ and $x\perp z$ hold in Riesz space E, then $x\perp(\alpha y + \beta z)$ holds for all $\alpha, \beta \in \mathbb{R}$.

(2) x and y are disjoint if and only if $|x + y| = |x - y|$.

(3) If $x\perp y$ holds in E, then $|x + y| = |x - y| = |x| + |y| = ||x| - |y|| = |x|\vee|y|$.

(4) Every subset of a Riesz space consisting of pairwise disjoint nonzero vectors is linearly independent.

Proof. (1) Let $x\perp y$, $x\perp z$ and $\alpha, \beta \in \mathbb{R}$. By the definition of the disjointness, we have $|x|\wedge|y| = 0$ and $|x|\wedge|z| = 0$. Then, we get

$$\begin{aligned}
0 &\leq |x| \wedge |\alpha y + \beta z| \\
&\leq |x| \wedge (|\alpha y| + |\beta z|) = |x| \wedge (|\alpha||y| + |\beta||z|) \\
&\leq |x| \wedge (|\alpha||y|) + (|x| \wedge (|\beta||z|) \\
&\leq (1 + |\alpha|)|x|\wedge(1 + |\alpha|)|y| + (1 + |\beta|)|x|\wedge(1 + |\beta|)|z| \\
&= (1 + |\alpha|)[|x|\wedge|y|] + (1 + |\beta|)[|x|\wedge|z|] \\
&= (1 + |\alpha|)0 + (1 + |\beta|)0 = 0.
\end{aligned}$$

So we found $|x| \wedge |\alpha y + \beta z| = 0$. This implies $x\perp(\alpha y + \beta z)$.

(2) Using the second identity in (14) of Theorem 1.1.2. we know $|x|\wedge|y| = \frac{1}{2}(|x + y| - |x - y|)$

is true in a Riesz space. From here, clearly $x \perp y$ are disjoint if and only if $|x + y| = |x - y|$. From here we infer that $x \perp y$ if and only if $|x + y| = |x - y|$ is true.

(3) Let If $x \perp y$ holds in Riesz space *E*. We know that $|x + y| = |x - y|$ is true by part (2). Let apply the same result to the vectors $|x|$ and $|y|$ then we get $||x| - |y|| = ||x| + |y|| = |x| + |y| = |x| \vee |y|$. Then notice that from the identity in part (13) of Theorem 1.1.2. yields
$|x + y| = |x - y| = |x| + |y| = ||x| - |y|| = |x| \vee |y|$.

(4) Let $x_1, \ldots, x_n$ be pairwise disjoint nonzero vectors in a Riesz space and $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$. By (1) and (3), we have $0 = |\alpha_1 x_1 + \ldots + \alpha_n x_n| = |\alpha_1 x_1| + \ldots + |\alpha_n x_n| = |\alpha_1||x_1| + \ldots + |\alpha_n||x_n|$. From here $|\alpha_i||x_i| = 0$ for every i. We know $|x_i| > 0$ for every i, so we get $|\alpha_i| = 0$ or $\alpha_i = 0$ for every $i = 1, \ldots, n$. As a result the nonzero vectors $x_1, \ldots, x_2$ are linearly independent. ■

Theorem 1.1.8 (The Riesz Decomposition Property). Let *E* be a Riesz space and the inequality $|x| \leq | y_1 + y_2 + \ldots + y_n |$ holds. Then there exist vectors $x_1, \ldots, x_n \in E$ satisfying $|x_i| \leq |y_i|$ for every $i = 1, \ldots, n$ and $x = x_1, \ldots, x_n$ .If the vector x is positive then the vectors x_i can be choosen positive.

Proof. Assume that $|x| \leq |y_1 + y_2|$. Then take $x_1 = [x \vee (-|y_1|)] \wedge |y_1|$. From the inequalities $-|y_1| \leq x \vee (-|y_1|)$ and $-|y_1| \leq |y_1|$, it follows that $-|y_1| \leq x_1$ or $-x_1 \leq |y_1|$. Moreover, from $x_1 \leq |y_1|$, we get $|x_1| = (-x_1) \vee x_1 \leq |x_1|$ (and if x positive then $0 \leq x_1 \leq x$ holds). Now take $x_2 = x - x_1$, then we get $x_2 = x - [x \vee (-|y_1|)] \wedge |y_1| = [0 \wedge (x + |y_1|)] \vee (x - |y_1|)$. On the other side , $|x| \leq |y_1| + |y_2|$ implies $-|y_1| - |y_2| \leq x \leq |y_1| + |y_2|$ which we obtain $-|y_2| = (-|y_2|) \wedge 0 \leq (x + |y_1|) \wedge 0 \leq x_2 \leq 0 \vee (x - |y_1|) \leq |y_2|$. Thus we get $|x_2| \leq |y_2|$. The general case can be proven by induction.■

Definition 1.1.9. Let *A* be a subset of a Riesz space *E*. *A* is called solid if $|x| \leq |y|$ for some $y \in A$ implies $x \in A$. If every subset *A* of *E* is contained in a smallest solid set, then it is called solid hull of *A* and indicated by Sol(*A*). From here, we see
$\mathrm{Sol}(A) = \{ y \in E: \exists x \in A \text{ such that } |y| \leq |x|\}$
Every solid set *A* is a balanced set if $x \in A$, then $\alpha x \in A$ for every $\alpha \in \mathbb{R}$ with $|\alpha| \leq 1$.
A subset *A* in a vector space is called convex if $\alpha x + (1 - \alpha)y \in A$ for all $x, y \in A$ and $0 \leq \alpha \leq 1$.

Definition 1.1.10. A is called a directed set if for $\alpha, \beta \in A$ there is a $\gamma \in A$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Definition 1.1.11. A net of a set X is a mapping $x : A \to X$ from a directed set A to X. $x(\alpha)$ will be denoted by x_α and the net $x : A \to X$ denoted by $\{x_\alpha\}$.

Definition 1.1.12. If $x_\alpha \leq x_\beta$ whenever $\alpha \leq \beta$ then a net $\{x_\alpha\}$ in a Riesz space is called increasing net(in symbols $x_\alpha\uparrow$). If $x_\beta \leq x_\alpha$ whenever $\alpha \leq \beta$ then a net $\{x_\alpha\}$ in a Riesz space is called decreasing net(in symbols $x_\alpha\downarrow$).

The notation $x_\alpha\uparrow x$ means that the net $\{x_\alpha\}$ is an inreasing net and supremum of the set $\{ x_\alpha : \alpha \in A\}$ exists and $\sup\{x_\alpha\} = x$. The notation $x_\alpha\downarrow x$ means that the net $\{x_\alpha\}$ is a decreasing net and infimum of the set $\{ x_\alpha : \alpha \in A\}$ exists and $\inf\{x_\alpha\} = x$.

Definition 1.1.13 (Order Convergence). A net $\{x_\alpha\}$ in a Riesz space E is order convergent to a element $x \in E$ ($x_\alpha \to x$) if there exists a set $\{y_\alpha\}$ of A such that $|x_\alpha - x| \leq y_\alpha\downarrow 0$. The element x is said to be the order limit of the net $\{x_\alpha\}$.

Lemma 1.1.14. A net in a Riesz space can have only one order limit.

Proof. Assume that the net $\{x_\alpha\}$ has two order limits in a Riesz space. Let $x_\alpha \to x$ and $x_\alpha \to t$. Using the definition of the order convergence, suppose that $\{y_\alpha\}$ and $\{z_\alpha\}$ be two nets satisfying $|x_\alpha - x| \leq y_\alpha\downarrow 0$ and $|x_\alpha - t| \leq z_\alpha\downarrow 0$. Then for every α we get $|x - t| \leq |x - x_\alpha| + |x_\alpha - t| \leq y_\alpha + z_\alpha$.
Since $y_\alpha + z_\alpha\downarrow 0$, it follows that $|x - t| = 0$, from here we get $x - t = 0$. So, $x = t$.■

Definition 1.1.15. A subset A of a Riesz space E is called order closed if $\{x_\alpha\} \subseteq A$ and $x_\alpha \to x$ in E that $x \in A$, then A is order closed if it contains its order limits. Similarly, if a subset A includes its sequential order limits, it is said to be σ-order closed.

Lemma 1.1.16. A solid subset A of a Riesz space E is order closed if and only if $\{x_\alpha\} \subseteq A$ and

$0 \le x_\alpha \uparrow x$ imply $x \in A$.

Proof. Assume that solid A satisfies the stated property and if $\{x_\alpha\} \subseteq A$ then $x_\alpha \to x$. Choose a net $\{y_\alpha\}$ satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \le y_\alpha$ for every α. From here we get $0 \le (|x| - y_\alpha)^+ \uparrow |x|$ and $(|x| - y_\alpha)^+ \le |x_\alpha|$ for every α. This implies $\{(|x| - y_\alpha)^+\} \subseteq A$. Hence we get $x \in A$.This indicates that A is order closed. ■

1.2. Ideals, Bands, and Riesz Subspaces

Definition 1.2.1. A solid subspace of a Riesz space *E* is called an ideal. A σ-order closed ideal is called a σ-ideal and an order closed ideal is called a band.
Let A and B be ideals, then their algebraic sum $A + B = \{ a + b: a \in A \text{ and } b \in B\}$ is an ideal too.

Lemma 1.2.2. An ideal *A* is a band if and only if $\{x_\alpha\} \subseteq A$ and $0 \le x_\alpha \uparrow x$ imply $x \in A$. Similarly, an ideal A is a σ-ideal if and only if $\{x_n\} \subseteq A$ and $0 \le x_n \uparrow x$ imply $x \in A$.

Definition 1.2.3. Let *A* be a nonempty subset of a Riesz space *E*. If *A* is included in a smallest ideal E_A, then it is called the ideal generated by *A*: $E_A = \{ x \in E: |x| \le \lambda\sum_{i=1}^{n}|x_i| \text{ such that } \exists x_1, \dots ,x_n \in A \text{ and } \lambda \ge 0\}$. A principal ideal of a Riesz space *E* is an ideal generated by a vector x. This ideal denoted by E_x : $E_x = \{ y \in E: |y| \le \lambda|x| \text{ such that } \exists\, \lambda \ge 0\}$.

Definition 1.2.4. Let E be a Riesz space. An element $0 < e \in E$ is called an order unit(or a strong unit) if for every $x \in E$ there is a $\lambda > 0$ such that $|x| \le \lambda e$.

Definition 1.2.5. Let *F* be a vector subspace of a Riesz space *E*. *F* is called a vector sublattice(or a Riesz subspace) if $x \vee y \in F$, $x \wedge y \in F$ whenever $x, y \in F$.

Let *F* be a Riesz subspace of a Riesz space *E*. If for every subset of *F* whose supremum(or infimum) exists in *F*, the supremum(or infimum) of the same subset exists in *E* and is the same as that in *F*, then the embedding of *F* into *E* preserves arbitrary suprema and infima.

Definition 1.2.6. Let F be a Riesz subspace of a Riesz space E. Then,

(1) F is called regular, if the embedding of F into E preserves arbitrary suprema and infima,

(2) F is called σ-regular, if the embedding of F into E preserves countable suprema and infima,

(3) F is called majorizing, if for every $x \in E$ there exists some $y \in E$ such that $x \leq y$.

Theorem 1.2.7. Let F be a Riesz subspace of a Riesz space E. Then, the following statements are equivalent.

(1) F is a regular subspace of E.

(2) If $\{x_\alpha\} \subseteq F$ satisfies $x_\alpha \downarrow 0$ in F, then $x_\alpha \downarrow 0$ hols in E.

(3) If $\{x_\alpha\} \subseteq F$ satisfies $x_\alpha \to x$ in F, then $x_\alpha \to x$ holds in E.

Lemma 1.2.8. Every ideal is a regular Riesz subspace.

Definition 1.2.9. Let F be a Riesz subspace of a Riesz space E. Then,

(1) F is called order dense in E, if for all $0 < x \in E$ $(0 \leq x, x \neq 0)$ there exists some $y \in F$ such that $0 < y \leq x$.

(2) F is called super order dense in E, if for all $0 < x \in E$ there exists a sequence $\{x_n\} \subseteq F$ with $0 \leq x_n \uparrow x$ in E.

From these definitions, every super order dense is order dense.

Theorem 1.2.10. Every order dense Riesz subspace of a Riesz space is a regular Riesz subspace.

Proof. Let F be a Riesz subspace of a Riesz space E. Then, assume that F is order dense in E and there is a net $\{x_\alpha\}$ satisfies $x_\alpha \downarrow 0$ in F. If $x_\alpha \downarrow 0$ does not hold in E, then there exist some $0 < x \in E$ with $0 < x \leq x_\alpha$ for every α. But since F is order dense in E, then from the definition there exists $y \in F$ such that $0 < y \leq x$ for all $0 < x \in E$. From here $0 < y \leq x_\alpha$ holds in F for every α, contradicting $x_\alpha \downarrow 0$ in F. Using Theorem 1.23., F is a regular Riesz subspace.■

Definition 1.2.11. Let A be a nonempty subset of a Riesz space E. The disjoint complement of A is defined by $A^d = \{ x \in E: x \perp y \ (|x| \wedge |y| = 0) \text{ for every } y \in A\}$.

Theorem 1.2.12. Let A be a nonempty subset of a Riesz space E. The disjoint complement of

A, A^d is an ideal.

Proof. Using the definition of ideal, we will show A^d is a solid subspace. Let x,y ∈ A^d then by the Definition 1.2.11. $|x|\wedge|z| = 0$ $|y|\wedge|z| = 0$ for every z ∈ A. From here, we know $0 \leq |x+y|\wedge|z| \leq (|x| + |y|)\wedge|z| \leq |x|\wedge|z| + |y|\wedge|z| = 0 + 0 = 0$. So, we get $|x+y|\wedge|z| = 0$ and this means x + y ∈ A^d. As a result A^d is a subspace. Then, let $|x| \leq |y|$ for some y ∈ A^d. If y ∈ A^d, then $|y|\wedge|z|=0$ for every z ∈ A. We have $0 \leq |x|\wedge|z| \leq |y|\wedge|z| = 0$. From here $|x|\wedge|z|=0$. This implies x ∈ A^d. Consequently by Definition 1.1.9. A^d is solid. Hence, A^d is an ideal.■

The disjoint complement $(A^d)^d$ is denoted by A^{dd}. It should be noted that $A \cap A^d = 0$ and $A \subseteq A^{dd}$. Moreover, if $A \subseteq B$ then $B^d \subseteq A^d$.

Theorem 1.2.13. Every ideal A of a Riesz space E is order dense in A^{dd}. In particular, an ideal A is order dense in E if and only if $A^d = \{0\}$.

Proof. The first proposition will be proved by contradiction. For this assume that ideal A is not order dense in A^{dd}. This means that there exists some $0 < x \in A^{dd}$ with no element y ∈ A satisfying $0 < y \leq x$. Since A is an ideal, we have $|y|\wedge|x| = 0$ for every y ∈ A. So, x ∈ A^d and from here $x \in A^d \cap A^{dd} = \{0\}$. This is a contradiction because x > 0. Consequently A is order dense in A^{dd}. Now for the second proposition let ideal A is order dense in E and $0 < x \in A^d$, then choose y ∈ A^d with $0 < y \leq x$ and from here we get $y \in A \cap A^d = \{0\}$. This is not possible because y > 0. So, if ideal A is order dense in E then, $A^d = \{0\}$. On the other hand let $A^d = \{0\}$, then $A^{dd} = E$ and therefore A is order dense in E. ■

Theorem 1.2.14. If A is an ideal of a Riesz space E, then the ideal $A \oplus A^d$ is order dense in E.

Proof. If $x \in (A \oplus A^d)$, then $x \in A^d \cap A^{dd} = \{0\}$. Hence, $(A \oplus A^d)^d = \{0\}$. From Theorem 1.2.13. we get ideal $A \oplus A^d$ is order dense in E.■

Definition 1.2.15. Let E be a Riesz space. E is called Archimedean if $\frac{1}{n}x\downarrow 0$ for every x ∈ E^+ and $n \in \mathbb{N}$ (if x, y ∈ E^+ and nx ≤ y for every n ∈ $\mathbb{N}$ imply x = 0).

Theorem 1.2.16. Let F be a Riesz subspace of an Archimedean Riesz space E. Then, the following statements are equivalent:

(1) *F* is order dense in *E*.

(2) For every $x \in E^+$ we have $x = \sup\{ y \in F: 0 \le y \le x\}$, or equally, for every $x \in E^+$ there exists a net $\{x_\alpha\} \subseteq F$ such that $0 \le x_\alpha \uparrow x$ in *E*.

Proof. (1) ⇒ (2) Let F be order dense in E. Assume, through contradiction, that $x = \sup\{ y\ x \in F: 0 \le y \le x\}$ is false. From here, there exists $0 < z \in E$ *such that* $y \in F$ *and* $0 \le y \le x$ imply $y \le x - z$. Choose some $a \in F$ such that $0 < a \le z$ and notice that $a \le z + (x - z) = x$. Then, $a \le x - z$. This means $2a = a + a \le (x - z) + z = x$. Due to induction, we get $0 < na \le x$ for n = 1, 2, …,. This contradicts with the Archimedean property of E. Thus, we get $x = \sup\{ y \in F: 0 \le y \le x\}$ for every $x \in E^+$.

(2) ⇒ (1) If for every $x \in E^+$, $x = \sup\{ y \in F: 0 \le y \le x\}$ holds in E, then clearly F is order dense in E. Because it is provides the definition of order dense. ■

Definition 1.2.17. Let *E* and *F* be Riesz spaces. An operator $T : E \to F$ is called linear operator if $T(x + y) = Tx + Ty$ and $T(\lambda x) = \lambda Tx$ for every $x, y \in E$ and $\lambda \in \mathbb{R}$.

A linear operator is $T : E \to F$ is called positive operator if $T(E^+) \subseteq F^+$ [or $Tx \ge 0$ whenever $x \ge 0$ in *E*].

Definition 1.2.18. A linear operator between Riesz spaces $T : E \to F$ is called:

(1) disjointness preserving if $x \perp y$ implies $Tx \perp Ty$ for all $x, y \in E$.

(2) interval preserving if $T[0,x] = [0,Tx]$ for all $x, y \in E^+$.

(3) a Riesz homomorphism (or a lattice homomorphism) if $T(x \vee y) = (Tx) \vee (Ty)$ and $T(x \wedge y) = (Tx) \wedge (Ty)$ for all $x, y \in E$.

(4) a Riesz σ-homomorphism (or a lattice σ-homomorphism) if *T* is a Riesz homomorphism and $x_n \to 0$ in *E* implies $T(x_n) \to 0$ in *F*.

(5) a normal Riesz homomorphism (or a normal lattice homomorphism) if *T* is a Riesz homomorphism and $x_\alpha \to 0$ in *E* implies $T(x_\alpha) \to 0$ in *F*.

Theorem 1.2.19. Every Riesz homomorphism (or a lattice homomorphism) is a positive operator.

Proof. Assume that $T : E \to F$ is called a lattice homomorphism. Let $x \geq 0$ in E. If $x \geq 0$ then $x = x \vee 0$. From here we get $Tx = T(x \vee 0)$. Using the lattice homomophism property of T we have $T(x \vee 0) = T(x) \vee (T0)$. Note that $T(0) = T(0 + 0) = T(0) + T(0) = 0$. So, $T(x) \vee (T0) = T(x) \vee 0 \geq 0$. This implies $Tx \geq 0$. From the Definition 1.2.17. T is positive operator.■

Theorem 1.2.20. Let $T : E \to F$ be a linear operator between Riesz spaces E and F. Then, the following statements are equivalent:

(1) T is a lattice homomorphism.

(2) $T(x^+) = (T(x))^+$ holds for all $x \in E$.

(3) $T(x \wedge y) = (Tx) \wedge (Ty)$ holds for all $x, y \in E$.

(4) If $x \wedge y = 0$, then $Tx \wedge Ty = 0$ for all $x, y \in E$.

(5) $|T(x)| = T(|x|)$ holds for all $x \in E$.

Proof. $1 \Rightarrow 2$ Using $x^+ = x \vee 0$ and $T(x \vee y) = (Tx) \vee (Ty)$

$T(x^+) = T(x \vee 0) = (Tx) \vee (T0) = (Tx) \vee (0) = (T(x))^+$

$2 \Rightarrow 3$ From the identity (9) in Theorem 1.1.2., we get $x + y = x \vee y + x \wedge y$

$x + y - x \vee y = x \wedge y$

$x + y - [-(-x) \wedge (-y)] = x \wedge y$

$x + y + (-x) \wedge (-y) = x \wedge y$

$x + (y-x) \wedge (0) = x \wedge y$

$x - (x-y) \vee 0 = x \wedge y$

$x - (x - y)^+ = x \wedge y$.

Then, $T(x \wedge y) = T(x - (x - y)^+) = T(x) - T(x - y)^+$ (linearity of T)

$= T(x) - (T(x-y))^+$ by (2)

$= Tx - (Tx - Ty)^+ = Tx - (Tx - Ty) \vee 0 = Tx - [-(Ty - Tx) \wedge 0] = Tx + (Ty - Tx) \wedge 0 = Ty \wedge Tx = Tx \wedge Ty$.

$3 \Rightarrow 4$ Let $x \wedge y = 0$. We know $T(0) = T(0 + 0) = T(0) + T(0) = 0$. From (3) we have $T(x \wedge y) = (Tx) \wedge (Ty)$. Clearly, $T(x \wedge y) = T(0) = 0 = (Tx) \wedge (Ty)$. So, if $x \wedge y = 0$, then $Tx \wedge Ty = 0$.

4 ⇒ 5 From the identity (10) in Theorem 1.1.2. we have $x = x^+ - x^-$ and $x^+ \wedge x^- = 0$. From (4), we get if $x^+ \wedge x^- = 0$, then $T(x^+) \wedge T(x^-) = 0$. Moreover, we will use the identities (12), (9) and (11) in Theorem 1.1.2. to prove this statement. So,

$|T(x)| = |T(x^+ - x^-)| = |T(x^+) - T(x^-)|$ linearity of T

$= T(x^+) \vee T(x^-) - T(x^+) \wedge T(x^-)$ by the identity (12) in Theorem 1.1.2.

$= T(x^+) \vee T(x^-) - 0$ by (4)

$= T(x^+) \vee T(x^-)$

$= T(x^+) + T(x^-) - T(x^+) \wedge T(x^-)$ by the identity (9) in Theorem 1.1.2.

$= T(x^+) + T(x^-)$

$= T(x^+ + x^-)$ linearity of T

$= T(|x|)$ by the identity (11) in Theorem 1.1.2.

5 ⇒ 1 From the first identity (8) in Theorem 1.1.2., we know $x \vee y = \frac{1}{2}(x + y + |x - y|)$. Then,

$T(x \vee y) = T[\frac{1}{2}(x + y + |x - y|)]$

$= \frac{1}{2} T[x + y + |x - y|]$ linearity of T

$= \frac{1}{2}[Tx + Ty + T|x - y|]$ linearity of T

$= \frac{1}{2}[Tx + Ty + |T(x - y)|]$ by (5)

$= \frac{1}{2}[Tx + Ty + |Tx - Ty|]$ linearity of T

$= Tx \vee Ty$ by the first identity (8) in Theorem 1.1.2.

From the (3) in Definition 1.2.18., T is a lattice homomorphism. ■

Definition 1.2.21. Let $T : E \rightarrow F$ be a linear operator between Riesz spaces E and F. The kernel of T is denoted by $Ker(T) = \{ x \in E: Tx = 0 \}$

Proposition 1.2.22. Let $T : E \rightarrow F$ be a lattice homomorphism between Riesz spaces E and F. The kernel of T is an ideal.

Proof. We know the kernel of T: $Ker(T) = \{ x \in E: Tx = 0 \}$. From the Definition 1.2.1., if we want to show that $Ker(T)$ is an ideal, then we need to show that $Ker(T)$ is a solid subspace. In that case, let $x, y \in Ker(T)$. Since $x, y \in Ker(T)$, $Tx = 0$ and $Ty = 0$. Then, $T(x + y) = Tx + Ty = 0 + 0 = 0$

implies $x + y \in Ker(T)$. After, let $\alpha \in \mathbb{R}$ and x, $\in Ker(T)$. Then, $T(\alpha x) = \alpha Tx = \alpha.0 = 0$ implies $\alpha x \in Ker(T)$. Consequently, *Ker(T)* is a subspace of *E*. After this, assume that $|x| \leq |y|$ and $y \in Ker(T)$. From $|x| \leq |y|$, we get $0 \leq |y| - |x|$

$$T(0) \leq T(|y| - |x|)$$

$$0 \leq T|y| - T|x| \qquad \text{(Linearity of } T\text{)}$$

$$T|x| \leq T|y|.$$

From the statement (5) in Theorem 1.2.20. we have $|T(x)| = T(|x|)$ and $|T(y)| = T(|y|)$. Then, $T|x| \leq T|y|$ means $|T(x)| \leq |T(y)|$. If $y \in Ker(T)$, then $Ty = 0$. So, $0 \leq |T(x)| \leq |T(y)| = 0$. This implies $|T(x)| = 0$ and then from here $Tx = 0$. So, $x \in Ker(T)$. Clearly, from the Definition 1.1.9. we see that *Ker(T)* is solid. Therefore, *Ker(T)* is an ideal. ■

Lemma 1.2.23. Let $T : E \to F$ be an onto Riesz homomorphism between Riesz spaces *E* and *F*. *T* is a normal Riesz homomorphism if and only if the kernel of *T* is a band of *E*. Likewise, *T* is a Riesz σ-homomorphism if and only if the kernel of *T* is a σ-ideal.

Proof. From the Proposition 1.2.22. we know that *Ker(T)* is an ideal. Suppose that *T* is a normal Riesz homomorphism and also a net $\{x_\alpha\} \subseteq Ker(T)$ satisfies $0 \leq x_\alpha \uparrow x$ in *E*. Since *T* is a normal Riesz homomorphism, we get $T(x) = \sup_\alpha T(x_\alpha) = \sup_\alpha 0 = 0$, and from here $x \in Ker(T)$. From the Lemma 1.2.2., *Ker(T)* is a band. For the opposite, suppose that *Ker(T)* is a band and also $0 \leq y \leq T(x_\alpha)$ holds in *F* for some y and for every α. And let $x_\alpha \downarrow 0$ in *E*. Since *T* is onto then, there exists some $z \in E$ with $T(z) = y$. Later, let $t_\alpha = (z^+ - x_\alpha)^+$ for every α and notice that $T(t_\alpha) = [T(z^+) – T(x_\alpha)]^+ = [y – T(x_\alpha)]^+ = 0$, so $t_\alpha \in Ker(T)$ for every α. Now, note that $0 \leq t_\alpha \uparrow z^+$, and using that *Ker(T)* is a band we get $z^+ \in Ker(T)$. Therefore, $y = T(z) = [T(z)]^+ = T(z^+) = 0$. This implies $T(x_\alpha) \downarrow 0$ in *F*. Consequently, from (5) in Definition 1.2.18 . *T* is a normal Riesz homomorphism. ■

Definition 1.2.24. A linear one-to-one lattice homomorphism between Riesz spaces is said to be a Riesz isomorphism (or a lattice isomorphism).

Definition 1.2.25. Let *E* and *F* be Riesz spaces. *E* and *F* are called Riesz isomorphic (or lattice isomorphic) if there exist a lattice isomorphism from *E* onto *F*.

Theorem 1.2.26. Let $T : E \to F$ be an onto Riesz homomorphism between Riesz spaces *E* and *F*.

Then T carries solid subsets of E to solid subsets of F.

Proof. Suppose that A is a solid subset of the Riesz space E and we have $|T(x)| \leq |T(y)|$ for some $y \in A$ and $x \in E$ in F. Put $z = [(-|y| \vee x] \wedge |y| \in E$ and notice that since $|z| \leq |y|$ holds then we get $z \in A$. Moreover, $T(z) = [(-|T(y)|) \vee T(x)] \wedge |T(y)| = T(x)$. So, $T(x) \in T(A)$ and thus $T(A)$ is a solid subset of F. ■

Theorem 1.2.27. Let $T : E \to F$ be a linear, one-to-one, onto operator between Riesz spaces Then, T is a Riesz isomorphism if and only if T and T^{-1} are positive operators.

Proof. ⇒ Assume that $T : E \to F$ is a lattice homomorphism and let $x \geq 0$ in E. Then, $x \geq 0$ implies $x = x \vee 0$. So, we get $Tx = T(x \vee 0) = Tx \vee T0 = Tx \vee 0 = (Tx)^+ \geq 0$. Since $Tx \geq 0$, then T is positive operator. Moreover, T is one-to-one and onto imply T has inverse and its denoted by $T^{-1}: F \to E$ with $y \to T^{-1}y$. Let $y \geq 0$ in F and there is a $x \in E$ such that $Tx = y$. From here, we get $T^{-1}(y) = T^{-1}(y \vee 0) = (T^{-1}y) \vee (T^{-1}0) = (T^{-1}y) \vee 0 = (T^{-1}y) \geq 0$. Since $T^{-1}(y) \geq 0$, then T^{-1} is positive operator.

⇐ Suppose that T and T^{-1} are positive operators. Remember that T is a Riesz homomorphism if and only if $T(x \vee y) = (Tx) \vee (Ty)$ for all $x, y \in E$. *Let* $x, y \in E$, *then* $x \leq x \vee y$ imply $T(x) \leq T(x \vee y)$ and $y \leq x \vee y$ imply $T(y) \leq T(x \vee y)$. From here, we get $T(x) \vee T(y) \leq T(x \vee y)$. For the other side, let $u, v \in F$. *Then,* $T^{-1}(u) \vee T^{-1}(v) \leq T^{-1}(u \vee v)$. We have $u = Tx$ for some $x \in E$ *and* $v = Ty$ *for some* $y \in E$. *Put* u *and* v *in* $T^{-1}(u) \vee T^{-1}(v) \leq T^{-1}(u \vee v)$. Then, we get

$T^{-1}(Tx) \vee T^{-1}(Ty) \leq T^{-1}(Tx \vee Ty)$

$Ix \vee Iy \leq T^{-1}(Tx \vee Ty)$

$x \vee y \leq T^{-1}(Tx \vee Ty)$

$T(x \vee y) \leq TT^{-1}(Tx \vee Ty)$

$T(x \vee y) \leq Tx \vee Ty$.

So, $T(x \vee y) = (Tx) \vee (Ty)$. This means T is a Riesz homomorphism. From Definition 1.2.24. T is a Riesz isomorphism. ■

1.3. Order Completeness and Riesz Algebra

Definition 1.3.1. Let A be a subset of a Riesz space E. A is called:

(1) bounded above if there is a $y \in E$ such that $x \leq y$ for each $x \in A$,

(2) bounded below if there is a $z \in E$ such that $z \leq x$ for each $x \in A$,

(3) bounded if A is both bounded above and below.

Definition 1.3.2. Let E be a Riesz space and for any two elements $x, y \in E$ with $x \leq y$, the set $[x,y] = \{ z \in E: x \leq z \leq y \}$ is said to be order interval(or interval). A subset A of Riesz space E is said to be order bounded if A is contained in an order interval.

Definition 1.3.3. A Riesz space E is called:

(1) Dedekind (or order) complete, if every nonempty subset of E which is bounded from above has a supremum or every nonempty subset of E which is bounded from below has an infimum.

(2) σ-Dedekind complete if every sequence that is bounded from above has a supremum.

Theorem 1.3.4. Let A be an order dense Riesz subspace of an Archimedean Riesz space E. A is an ideal of E if A is Dedekind complete in its own right.

Proof. Suppose that $0 \leq x \leq y$ with $x \in E$ and $y \in A$. Because A is order dense in E and E is Archimedean, by Theorem 1.2.16. there exists a net $\{x_\alpha\}$ of A with $0 \leq x_\alpha \uparrow x$ in E. Then, A is Dedenkind complete, therefore $0 \leq x_\alpha \uparrow z$ holds in A for some $z \in A$. From Theorem 1.2.10. , A is a regular Riesz subspace of E and so $x_\alpha \uparrow z$ holds also in E. Then, $x = z \in A$. Thus A is an ideal of E.■

Definition 1.3.5. Let A be a Riesz space(or vector lattice). A is said to be a Riesz algebra(lattice ordered algebra) if it has an associative multiplication and it is an algebra and additively $x \geq 0$ and $y \geq 0$ in A imply $xy \geq 0$ in A.

Definition 1.3.6. Let A be a lattice ordered algebra. Then, A is called:

(1) an f-algebra if $x \wedge y = 0$ in A, then $ax \wedge y = xa \wedge y$ for all $0 \leq a \in A$,

(2) an almost f-algebra if $x \wedge y = 0$ in A, then $xy = 0$ in A,

(3) a d-algebra if $x \wedge y = 0$ in A, then $ax \wedge ay = 0 = xa \wedge ya$ for all $0 \leq a \in A$.

Proposition 1.3.7. Let A be an f-algebra and x, y $\in A$. If $x \perp y$, then $xy = 0$.

Proof. From the Definition 1.1.6. we know that $x \perp y$ imply $|x| \wedge |y| = 0$. Let $x \wedge y = 0$, then we know $0 = x \wedge y \leq x$ and $0 = x \wedge y \leq y$. A is an f-algebra so we get

$yx \wedge y = 0$ (f-algebra and $y \geq 0$)

$y \wedge yx = 0$ (commutative)

$yx \wedge yx = 0$ (f-algebra and $x \geq 0$). This implies $yx = 0$. In addition,

$xy \wedge y = 0$ (f-algebra and $y \geq 0$)

$y \wedge xy = 0$ (commutative)

$xy \wedge xy = 0$ (f-algebra and $x \geq 0$). This implies $xy = 0$.We know that, $x^+ \leq |x|$, $x^- \leq |x|$, $y^+ \leq |y|$ and $y^- \leq |y|$. From here,

$0 \leq x^+ \wedge y^+ \leq |x| \wedge |y| = 0$ implies $x^+ \wedge y^+ = 0$

$0 \leq x^+ \wedge y^- \leq |x| \wedge |y| = 0$ implies $x^+ \wedge y^- = 0$

$0 \leq x^- \wedge y^+ \leq |x| \wedge |y| = 0$ implies $x^- \wedge y^+ = 0$

$0 \leq x^- \wedge y^- \leq |x| \wedge |y| = 0$ implies $x^- \wedge y^- = 0$. Using the $x \wedge y = 0$ implies $xy = 0$, we get $x^+y^+ = 0$, $x^+y^- = 0$, $x^-y^+ = 0$ and $x^-y^- = 0$. From the first identity (10) in Theorem 1.1.2. we have $x = x^+ - x^-$ and $y = y^+ - y^-$. So, $xy = (x^+ - x^-)(y^+ - y^-) = x^+y^+ - x^+y^- - x^-y^+ + x^-y^- = 0.$ ■

Proposition 1.3.8. Let A be a f-algebra and $x \in A$. Then, $xx = x^2 \geq 0$.

Proof. From the identity (10) in Theorem 1.1.2. we know $x = x^+ - x^-$ and $x^+ \wedge x^- = 0$. So, using the proposition 1.3.7., $x^+ \wedge x^- = 0$ implies $x^+x^- = 0$ and $x^- \wedge x^+ = 0$ implies $x^-x^+ = 0$. Moreover, we know

$0 \leq x^+$ and $0 \leq x^+$ imply $0 \leq x^+x^+ = (x^+)^2$

$0 \leq x^-$ and $0 \leq x^-$ imply $0 \leq x^-x^- = (x^-)^2$.

Note that,

$x^2 = xx = (x^+ - x^-)(x^+ - x^-) = x^+x^+ - x^+x^- - x^-x^+ + x^-x^- = (x^+)^2 + (x^-)^2 \geq 0.$ ■

RESULTS

First, we described and proved lattice identities. Then, we studied some lemmas and theorems of Riesz spaces. The subject of Riesz spaces, of which we have shown only a small part, is very broad and is used in many different sciences apart from mathematics like economy.

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